An operator approach to the construction of generating functions for products of associated Laguerre polynomials

1996 J. Phys. A: Math. Gen. 29 L263

(http://iopscience.iop.org/0305-4470/29/10/007)

J. Phys. A: Math. Gen. **29** (1996) L263–L270. Printed in the UK

LETTER TO THE EDITOR

An operator approach to the construction of generating functions for products of associated Laguerre polynomials

A Messina and E Paladino
INFM-GNSM and Istituto di Fisica dell'Università di Palermo, Via Archirafi 36, 90123 Palermo, Italy

Received 14 September 1995

Abstract. We introduce an iteratively-defined operator sequence allowing the construction of new useful mathematical identities, involving the product of associated Laguerre polynomials. Possible physical applications as well as some methodological aspects of our approach are pointed out.

1. Introduction

In this letter we wish to present an operator method which allows the derivation of new mathematical identities involving the product of associated Laguerre polynomials. The possibility of obtaining such a result emerged while investigating the properties of the ground state of a two-level system linearly coupled to a set of quantum harmonic oscillators [1]. Attempting, in fact, to determine the exact expression for the ground state of this system taking the zero temperature limit of its free energy, we were led to focus our attention on a particular sequence of appropriate operators admitting an iterative definition. Such operators have attracted our interest since they possess two basic features which may be directly related to their structure. The first one is the possibility of identifying a convenient basis wherein the trace of each operator in the sequence exists and is expressed by a series, that is by a non-closed form. The second one is the existence of another orthogonal basis, easily brought to light, wherein, once more, the trace of an arbitrary operator in the sequence is convergent. This differs from the other basis, however, since in this case the trace may be given in a closed form. Our approach exploits the knowledge of these operators, taking advantage of the fact that their traces may be evaluated using two different bases. The relations of interest that we wish to establish follow by simply equating the two constructed expressions of the trace.

This letter is organized as follows. Section 2 is devoted to the presentation of the sequence of operators. The two expressions of the trace of an arbitrary operator in the sequence are deduced in sections 3 and 4. The new mathematical identities are presented and discussed in section 5. Finally, in section 6, we point out some methodological aspects of our treatment.

2. Definition of the operator A_p

Consider the following operator

$$A_0 = \rho(\beta_0) = \exp(-\beta_0 H_0) \tag{1}$$

0305-4470/96/100263+08$19.50

with $\beta_0 \in C$ and

$$H_0 = \alpha^\dagger\alpha + \lambda^*\alpha + \lambda\alpha^\dagger \tag{2}$$

where α and $\alpha^\dagger$ are bosonic annihilation and creation operators obeying the commutation relation $[\alpha, \alpha^\dagger] = 1$.

As is evident from its definition, H_0 represents the Hamiltonian of a displaced harmonic oscillator of unitary frequency and complex displacement parameter λ.

It is well known that putting $\alpha^\dagger\alpha|n\rangle = n|n\rangle$, we get

$$H_0|\psi_n\rangle = E_n|\psi_n\rangle \tag{3}$$

where

$$|\psi_n\rangle = D^\dagger(\lambda)|n\rangle \tag{4}$$

and

$$E_n = n - |\lambda|^2. \tag{5}$$

The unitary displacement operator $D(a)$ $(a \in C)$ has the form [2]

$$D(a) = \exp\{a\alpha^\dagger - a^*\alpha\} = D^\dagger(-a) \tag{6}$$

and, acting on the vacuum state, generates the coherent state $|a\rangle$

$$|a\rangle = D(a)|0\rangle = \mathrm{e}^{-|a|^2/2}\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}|n\rangle. \tag{7}$$

Equation (4) means that this operator accomplishes the canonical diagonalization of H_0, being

$$D(\lambda)H_0D^\dagger(\lambda) = \alpha^\dagger\alpha - |\lambda|^2. \tag{8}$$

It is easy to convince oneself that equation (8) makes it simple to evaluate the trace of A_0 in the coherent state basis. On the other hand, the same trace can be calculated immediately, using the complete set of eigenstates of H_0. In both cases we get the same closed expression:

$$\mathrm{Tr}(A_0) = \frac{\mathrm{e}^{\beta_0|\lambda|^2}}{1 - \mathrm{e}^{-\beta_0}} \tag{9}$$

provided that $\mathrm{Re}(\beta_0) > 0$. Thus, calculating the trace of the simple operator A_0 using two different bases does not generate any formula. On the contrary, consider the following iteratively-defined sequence of operators:

$$A_p = A_{p-1}\cos(\pi\alpha^\dagger\alpha)\rho(\beta_p) \qquad (p \geqslant 1) \tag{10}$$

with $\beta_i \in C$ $\forall i \in N$. It is not difficult to convince oneself that for any $p \geqslant 1$, the trace of A_p in the $\{|\psi_n\rangle\}$ basis acquires the structure of a complex series (multiple if $p \geqslant 1$) whose coefficients are matrix elements of the same operator $\cos(\pi\alpha^\dagger\alpha)$. On the other hand, the specific structure of A_p, related to the iterative definition (10), assures the possibility of evaluating the trace of each A_p, even using the coherent basis. We shall prove that for every p the calculation of the trace in such a basis leads to a closed expression. In this way, exploiting the well known independence of the trace of an arbitrary operator (provided it exists) on the basis used for its calculation [3], we immediately get a class of mathematical identities.

3. The trace of A_p in the $\{|\psi_n\rangle\}$ basis

Using equation (10), the trace of A_p in the basis of the eigenstates of H_0 may be expressed as

$$\mathrm{Tr}(A_p) = \sum_{s_0=0}^{\infty}\langle\psi_{s_0}|\rho(\beta_0)\cos(\pi\alpha^{\dagger}\alpha)\rho(\beta_1)\cos(\pi\alpha^{\dagger}\alpha)\cdots\cos(\pi\alpha^{\dagger}\alpha)\rho(\beta_p)|\psi_{s_0}\rangle$$
$$= \sum_{s_0=0}^{\infty}\mathrm{e}^{-(\beta_0+\beta_p)E_{s_0}}\sum_{s_1=0}^{\infty}\mathrm{e}^{-\beta_1 E_{s_1}}\cdots\sum_{s_{p-1}=0}^{\infty}\mathrm{e}^{-\beta_{p-1}E_{s_{p-1}}}\langle\psi_{s_0}|\cos(\pi\alpha^{\dagger}\alpha)|\psi_{s_1}\rangle$$
$$\cdots\langle\psi_{s_{p-1}}|\cos(\pi\alpha^{\dagger}\alpha)|\psi_{s_0}\rangle. \tag{11}$$

Since

$$\cos(\pi\alpha^{\dagger}\alpha)D^{\dagger}(\lambda) = D(\lambda)\cos(\pi\alpha^{\dagger}\alpha) = D(\lambda)\mathrm{e}^{\mathrm{i}\pi\alpha^{\dagger}\alpha} \tag{12}$$

we get [2]

$$\langle\psi_{J'}|\cos(\pi\alpha^{\dagger}\alpha)|\psi_J\rangle = (-1)^J\langle J'|D(2\lambda)|J\rangle$$
$$= (-1)^J\left(\frac{J!}{J'!}\right)^{1/2}(2\lambda)^{J'-J}\,\mathrm{e}^{-|2\lambda|^2/2}L_J^{(J'-J)}(|2\lambda|^2). \tag{13}$$

The associated Laguerre polynomials $L_J^{(\alpha)}(x)$ can be explicitly expressed as

$$L_J^{(\alpha)}(x) = \sum_{l=\max(0,-\alpha)}^{J}(-1)^l\binom{J+\alpha}{J-l}\frac{x^l}{l!} \tag{14}$$

with $x \geqslant 0$ and $\alpha + J \geqslant 0$, $J \geqslant 0$ [4].

Inserting equation (13) into equation (11) we obtain the following non-closed expression of $\mathrm{Tr}(A_p)$:

$$\mathrm{Tr}(A_p) = N_p\sum_{s_0,s_1,\ldots,s_{p-1}=0}^{\infty}L_{s_1}^{(s_0-s_1)}(x)L_{s_2}^{(s_1-s_2)}(x)\cdots L_{s_0}^{(s_{p-1}-s_0)}(x)z_0^{s_0}z_1^{s_1}\cdots z_{p-1}^{s_{p-1}} \tag{15}$$

where

$$x = |2\lambda|^2 \qquad z_0 = -\mathrm{e}^{-(\beta_0+\beta_p)} \qquad z_J = -\mathrm{e}^{-\beta_J} \qquad J = 1,\ldots,p-1 \tag{16}$$

and

$$N_p = [(-1)^p z_0 z_1\cdots z_{p-1}]^{-x/4}\mathrm{e}^{-px/2}. \tag{17}$$

Of course equation (15) is meaningful only if we show that the convergence radius $\bar{r}$ of the polycircle wherein the multiple complex power series converges is strictly positive [5]. Observing that

$$|L_J^{(J'-J)}(x)| \leqslant \sum_{l=\max\{0,J-J'\}}^{J}\binom{J'}{J-l}\frac{x^l}{l!} \leqslant \left[\sum_{l=0}^{J'}\binom{J'}{l}\right]\left[\sum_{l=0}^{J}\frac{x^l}{l!}\right] \leqslant 2^{J'}\mathrm{e}^x \tag{18}$$

we immediately have

$$|L_{s_1}^{(s_0-s_1)}(x)|\cdots|L_{s_0}^{(s_{p-1}-s_0)}(x)|\,|z_0|^{s_0}\cdots|z_{p-1}|^{s_{p-1}} \leqslant |2z_0|^{s_0}|2z_1|^{s_1}\cdots|2z_{p-1}|^{s_{p-1}}\mathrm{e}^{px}. \tag{19}$$

Inequality (19) assures the absolute convergence of $\mathrm{Tr}(A_p)$ inside the polycircle $K(r,0) \subset C^p$ centred at the origin of C^p and having radius $r = (r_0 = \frac{1}{2}, r_1 = \frac{1}{2},\ldots,r_{p-1} = \frac{1}{2})$. In other words, the series is absolutely convergent in the set of points $(z_0, z_1,\ldots,z_{p-1}) \in C^p$ satisfying the conditions $|z_J| < r_J$, $\forall J = 0, 1,\ldots,p-1$. Obviously, inequality (18) does not determine the convergence radius $\bar{r}$ of series (15). In fact, if $\bar{K}(\bar{r}, 0)$ is the convergence polycircle of (15), the condition $K(r, 0) \subseteq \bar{K}(\bar{r}, 0)$ holds. Generally speaking, finding the sum of a series like (15) is very difficult. However, if we are able to show that the trace of A_p can be expressed in closed form using another basis, then we succeed in summing the series (15). In the next section we show that this is the case with the help of the coherent basis.

4. The trace of A_p in the coherent basis

The completeness of the coherent states allows one to write the trace of the operator A_p in the form

$$\mathrm{Tr}[A_p] = \int \frac{\mathrm{d}^2 a}{\pi} \langle a|A_p|a\rangle \tag{20}$$

where the integration is over all the complex plane. Exploiting the operator identity, the immediate consequence of equation (8),

$$\rho(\beta) = \mathrm{e}^{\beta|\lambda|^2} D^\dagger(\lambda)\mathrm{e}^{-\beta\alpha^\dagger\alpha} D(\lambda) \tag{21}$$

and taking into consideration equation (12) and that [2]

$$D(\lambda)|a\rangle = \exp\{\tfrac{1}{2}[\lambda a^* - \lambda^* a]\}|a + \lambda\rangle \tag{22}$$

the expression of $\langle a|A_p|a\rangle$ can be cast in the form

$$\langle a|A_p|a\rangle = \mathrm{e}^{\beta^{(0)}|\lambda|^2}\langle a + \lambda|\left(\prod_{i=1}^{p/2} B_i\right)\mathrm{e}^{-\beta_p\alpha^\dagger\alpha}|a + \lambda\rangle \tag{23}$$

where p is a positive even integer,

$$\beta^{(0)} = \sum_{t=0}^{p} \beta_t \tag{24}$$

and

$$B_i = \mathrm{e}^{-\beta_{2(i-1)}\alpha^\dagger\alpha} D(2\lambda)\mathrm{e}^{-\beta_{2i-1}\alpha^\dagger\alpha} D^\dagger(2\lambda). \tag{25}$$

When p is an odd positive integer, $\langle a|A_p|a\rangle$ becomes

$$\langle a|A_p|a\rangle = \langle a + \lambda|\left(\prod_{i=1}^{(p-1)/2} B_i\right) C_p \mathrm{e}^{\mathrm{i}\pi\alpha^\dagger\alpha}|a + \lambda\rangle \mathrm{e}^{|\lambda|^2\beta^{(0)}} \tag{26}$$

where

$$C_p = \mathrm{e}^{-\beta_{p-1}\alpha^\dagger\alpha} D(2\lambda)\mathrm{e}^{-\beta_p\alpha^\dagger\alpha}. \tag{27}$$

Equation (26) maintains its validity for $p = 1$ too, provided we consistently put $\prod_{i=1}^{0} B_i$ equivalent to the identity operator.

Using the well known transformation property [6]

$$\exp(x\alpha^\dagger\alpha) F(\alpha, \alpha^\dagger) \exp(-x\alpha^\dagger\alpha) = F(\alpha\mathrm{e}^{-x}, \alpha^\dagger\mathrm{e}^{x}) \tag{28}$$

with $x \in C$, we succeed in putting expressions (23) and (26) in the following more convenient form:

$$\langle a|A_p|a\rangle = \langle a + \lambda|\mathrm{e}^{-\beta^{(0)}\alpha^\dagger\alpha}\left(\prod_{i=1}^{p} D_i\right)\mathrm{e}^{\mathrm{i}p\pi\alpha^\dagger\alpha}|a + \lambda\rangle \mathrm{e}^{|\lambda|^2\beta^{(0)}} \tag{29}$$

where

$$D_i = \exp\{2(-1)^{i-1}[\lambda e^{\beta^{(i)}}\alpha^\dagger - \lambda^* e^{-\beta^{(i)}}\alpha]\} \tag{30}$$

with

$$\beta^{(i)} = \sum_{t=i}^{p} \beta_t. \tag{31}$$

The product of exponential operators D_i appearing in equation (29) may be converted into a single exponential operator exploiting the following identity:

$$\prod_{i=1}^{p} e^{(x_i\alpha^\dagger - y_i\alpha)} = e^{X\alpha^\dagger - Y\alpha} e^{Z/2} \tag{32}$$

where

$$X = \sum_{i=1}^{p} x_i \qquad x_i \in C \tag{33}$$

$$Y = \sum_{i=1}^{p} y_i \qquad y_i \in C \tag{34}$$

$$Z = \sum_{\substack{i,j=1 \\ (i<j)}}^{p} \{x_i, y_j\} \tag{35}$$

with

$$\{x_i, y_j\} = (x_i y_j - x_j y_i). \tag{36}$$

Equation (32) may be proved by induction, using the Glauber identity [3]. With the help of equation (32) and taking into account that

$$e^{X\alpha^\dagger - Y\alpha} = e^{X\alpha^\dagger} e^{-Y\alpha} e^{-XY/2} \tag{37}$$

the mean value of A_p in the coherent state $|a\rangle$ may be written as

$$\langle a|A_p|a\rangle = e^{|\lambda|^2\beta^{(0)}} \langle a+\lambda| e^{-\beta^{(0)}\alpha^\dagger\alpha} e^{\tilde{X}\alpha^\dagger} e^{-\tilde{Y}\alpha} e^{i p\pi\alpha^\dagger\alpha} |a+\lambda\rangle e^{-(\tilde{X}\tilde{Y}-\tilde{Z})/2} \tag{38}$$

where $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are functions of $\{\beta_i\}$ and λ explicitly given by

$$\tilde{X} = 2\lambda \sum_{i=1}^{p} (-1)^{i-1} e^{\beta^{(i)}} \tag{39}$$

$$\tilde{Y} = 2\lambda^* \sum_{i=1}^{p} (-1)^{i-1} e^{-\beta^{(i)}} \tag{40}$$

$$\tilde{Z} = 4|\lambda|^2 \sum_{\substack{i,j=1 \\ (i<j)}}^{p} (-1)^{i+j} \{e^{\beta^{(i)}}, e^{-\beta^{(j)}}\}. \tag{41}$$

Using equation (28), we finally get

$$\langle a|A_p|a\rangle = A \exp\{g|a|^2 + (\tilde{X} e^{-\beta^{(0)}} + \lambda g)a^* - [(-1)^p \tilde{Y} - g\lambda^*]a\} \tag{42}$$

where

$$g = [e^{-(\beta^{(0)} - i\pi p)} - 1] \tag{43}$$

and

$$A \equiv A(\{\beta_i\}, \lambda, p) = \mathrm{e}^{g|\lambda|^2}\mathrm{e}^{|\lambda|^2\beta^{(0)}}\mathrm{e}^{-(\tilde{X}\tilde{Y}-\tilde{Z})/2}\mathrm{e}^{[(-1)^{p+1}\lambda\tilde{Y}+\mathrm{e}^{-\beta^{(0)}}\lambda^*\tilde{X}]}. \tag{44}$$

To deduce equation (42) we have used the expression of the mean value of the operator $\exp(-x\alpha^\dagger\alpha)$ on the coherent state $|a+\lambda\rangle$ [6],

$$\langle a+\lambda|\mathrm{e}^{-x\alpha^\dagger\alpha}|a+\lambda\rangle = \exp\{|a+\lambda|^2(\mathrm{e}^{-x}-1)\}. \tag{45}$$

Inserting the expression of $\langle a|A_p|a\rangle$, as given by equation (42), into equation (20), yields an integral over all the complex plane that converges if $\mathrm{Re}(g) < 0$ [2]. It is convenient to express such a condition in terms of the complex variables $\{z_i\}$, as given by equation (16), obtaining for any positive integer

$$\mathrm{Re}[z_0 z_1 \ldots z_{p-1} - 1] < 0. \tag{46}$$

When inequality (46) is satisfied, the trace of A_p in the coherent basis exists and can be expressed in closed form as follows,

$$\mathrm{Tr}[A_p] = -\frac{A}{g}\exp\left\{\frac{(\tilde{X}\mathrm{e}^{-\beta^{(0)}} + \lambda g)((-1)^p\tilde{Y} - g\lambda^*)}{g}\right\} \tag{47}$$

or, more simply, as

$$\mathrm{Tr}[A_p] = -\frac{1}{g}\mathrm{e}^{|\lambda|^2\beta^{(0)}}\exp\left\{-\frac{1}{2}(\tilde{X}\tilde{Y}-\tilde{Z}) + (-1)^p\frac{\mathrm{e}^{-\beta^{(0)}}\tilde{X}\tilde{Y}}{g}\right\}. \tag{48}$$

It is easy to convince oneself that whatever the positive integer p, $\mathrm{Tr}(A_p)$ can be explicitly given as a function of the p complex variables $\{z_i\}$, using equations (16), (24), (31), (39), (40), (41), (43) and (44).

5. Generating function for the product of p associated Laguerre polynomials

Summing up, we have shown that the trace of the operator A_p may be evaluated using two different bases. We remark that the polycircle in C^p wherein the convergence of the multiple complex series (15) has been proved lies inside the region of C^p characterized by inequality (46). Thus, at least in this polycircle of C^p, the two expressions of the trace of A_p must coincide. In this way we construct a class of relations which represent the main result of our paper. Comparing equation (15) with equation (47) we in fact get

$$\sum_{s_0,s_1,\ldots,s_{p-1}=0}^{\infty} L_{s_1}^{(s_0-s_1)}(x)L_{s_2}^{(s_1-s_2)}(x)\cdots L_{s_0}^{(s_{p-1}-s_0)}(x)z_0^{s_0}z_1^{s_1}\cdots z_{p-1}^{s_{p-1}}$$
$$= -\frac{\mathrm{e}^{px/2}}{g}\exp\left\{-\frac{1}{2}(\tilde{X}\tilde{Y}-\tilde{Z}) + \frac{(-1)^p\tilde{X}\tilde{Y}\mathrm{e}^{-\beta^{(0)}}}{g}\right\}. \tag{49}$$

It is useful to consider some particular cases of equation (49). Assuming $p = 1$, it is possible to deduce the following well known formula for the generating function of the Laguerre polynomials [4]:

$$\sum_{s_0=0}^{\infty} L_{s_0}(x)z_0^{s_0} = \frac{1}{1-z_0}\exp\left\{-x\frac{z_0}{1-z_0}\right\}. \tag{50}$$

We write down explicitly two other particular cases putting $p = 2$ and $p = 3$ in equation (49). In this way we achieve the following results which, to the best of our knowledge, are not explicitly present in the literature:

$$\sum_{s_0=0}^{\infty}\sum_{s_1=0}^{\infty} L_{s_1}^{(s_0-s_1)}(x) L_{s_0}^{(s_1-s_0)}(x) z_0^{s_0} z_1^{s_1} = \frac{1}{1-z_0 z_1} \exp\left\{-x \frac{2z_0 z_1 + [z_0 + z_1]}{1 - z_0 z_1}\right\} \tag{51}$$

$$\sum_{s_0=0}^{\infty}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty} L_{s_1}^{(s_0-s_1)}(x) L_{s_2}^{(s_1-s_2)}(x) L_{s_0}^{(s_2-s_0)}(x) z_0^{s_0} z_1^{s_1} z_2^{s_2}$$
$$= \frac{1}{1-z_0 z_1 z_2} \exp\left\{-x \frac{3z_0 z_1 z_2 + [z_0(z_1+z_2) + z_1 z_2] + [z_0 + z_1 + z_2]}{1 - z_0 z_1 z_2}\right\}. \tag{52}$$

Of course, continuing with greater values of the integer p, we may obtain without difficulty, other relations involving the product of p appropriately related associated Laguerre polynomials.

6. Physical applications and conclusive remarks

An interesting possible physical application of the mathematical results obtained in the previous sections is the calculation of the free energy for a spin-boson Hamiltonian model [7]. If, in fact, in the context of this problem, an appropriate Dyson expansion of the statistical operator is considered, the identity (49) deduced in this paper helps in obtaining closed expressions for the trace of a generic term in the Dyson operator series. Carrying this procedure further should make it easier to handle the evaluation of the partition function of the system.

We wish to conclude by pointing out that the success of the approach followed in this letter is strongly related to the structure of the operator A_p. Such an operator, in fact, has been chosen in such a way as to make it easy to single out two different bases wherein the evaluation of its trace may be brought to an end. As far as the form of H_0, we remark that, in view of equation (12), it is not difficult to understand why and how the operator $\cos(\pi\alpha^\dagger\alpha)$ is present in A_p. To this end it is enough to consider that, alternating, in the expression of A_p, the operators $\rho(\beta_i)$ with the operator $\cos(\pi\alpha^\dagger\alpha)$, the trace of A_p in the $\{|\psi_n\rangle\}$ basis contains p different but related matrix elements of $D(2\lambda)$ between number states.

The knowledge of the generating function of a product of associated Laguerre polynomials, as expressed by equation (49), may be usefully applied to those physical [8–13] or chemical [14–16] contexts where displaced number states of an harmonic oscillator appear.

References

[1] Paladino E 1994 Caratterizzazione Fisica dello Stato Fondamentale dell'Interazione Spin-Bosone in Regime di Accoppiamento Intermedio *Thesis* Università degli Studi di Palermo
[2] Cahill K E and Glauber J 1969 *Phys. Rev.* **177** 1857
[3] Cohen-Tannoudji C, Diu B and Laloe F 1977 *Quantum Mechanics* (New York: Wiley) pp 166, 174
[4] Szego G 1993 *Orthogonal Polynomials* (Providence, RI: American Mathematical Society) 4th edn pp 100, 101
[5] Sveshnikov A and Tikhonov A 1978 *The Theory of Functions of a Complex Variable* (Moscow: MIR) 2nd edn p 310
[6] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley) pp 154, 156

[7] Benivegna G, Messina A and Paladino E 1996 Ground state symmetry classification for a non-isolated tunneling particle *J. Phys. A: Math. Gen.* in press

[8] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 *Rev. Mod. Phys.* **59** 1

[9] Bishop R F and Voudras A 1994 *Phys. Rev.* A **50** 4488

[10] Benivegna G and Messina A 1987 *Phys. Rev.* A **35** 3313
Stolze J and Muller L 1990 *Phys. Rev.* B **42** 6704

[11] Hang Z 1989 *J. Phys. C: Solid State Phys.* **1** 1641
Lo C F and Sollie R 1993 *Phys. Rev.* B **48** 10 183
Kandemir B S and Altanhan T 1994 *J. Phys. C: Solid State Phys.* **6** 4505
Qinghu C, Kelin W and Shaolong W 1994 *Phys. Rev.* B **50** 164

[12] Janszky J, Vinogradov A V, Walmsley I A and Mostowski J 1994 *Phys. Rev.* A **50** 732

[13] De Oliveira F A M, Kim M S, Knight P L and Buzek V 1990 *Phys. Rev.* A **42** 2645
Brisudova M 1991 *J. Mod. Opt.* **38** 2505

[14] Benderskii V A, Makarov D E and Wight C A 1993 Chemical dynamics at low temperatures *Advances in Chemical Physics* vol LXXXVIII (New York: Wiley)

[15] Silbey R and Harris R A 1989 *J. Phys. Chem.* **93** 7062

[16] Silbey R 1991 *Tunneling and Relaxation in Low Temperature Systems in Large-Scale Molecular Systems* ed W Gans *et al* (New York: Plenum)